When is a function not a function?

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When is a function not a function?

- When it’s a number
- When it’s a vector
- When it’s a point

Early on we learn. . .

“Functions are things that map numbers to other numbers”

But we also know...

There are transformations of functions into other functions;

$$x^2 \longrightarrow \frac{1}{3}x^3$$

$$\ln(x) \longrightarrow x\ln(x) - x$$

and, in general,

$$f \longrightarrow \int f$$

or, put more carefully,

$$f \longrightarrow g, \text{ where } g(x) = \int_{x_0}^{x} f(t)\,dt$$

Another Example: Differentiation

$$f \longrightarrow f'$$

For example,

$$x^2 \longrightarrow 2x$$

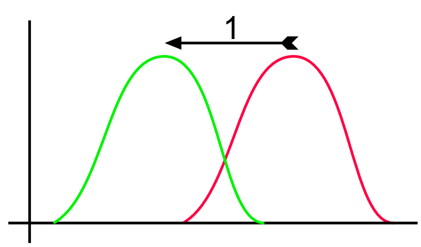
or,

$$\sin(x) \longrightarrow \cos(x)$$

Another Example: Shifting

$$f \longrightarrow g, \text{ where } g(x) := f(x+1)$$

Illustrated by a picture like;



Moral

So, there are “functions” that map ordinary functions into other ordinary functions.

These “super-functions” are usually called *operators*.

What's the domain of an operator?

Typically operators act on vector spaces of functions.

For example:

- $C[a,b]$ is the set of all continuous functions on the interval $[a,b]$.

- $L^2[a,b]$ is the set of all functions on $[a,b]$ where $\int_a^b |f(t)|^2 \, dt$ exists and is finite.

Norms on Vector Spaces

If functions are like vectors then we can also think of them as being like points in space. And measure their distance apart.

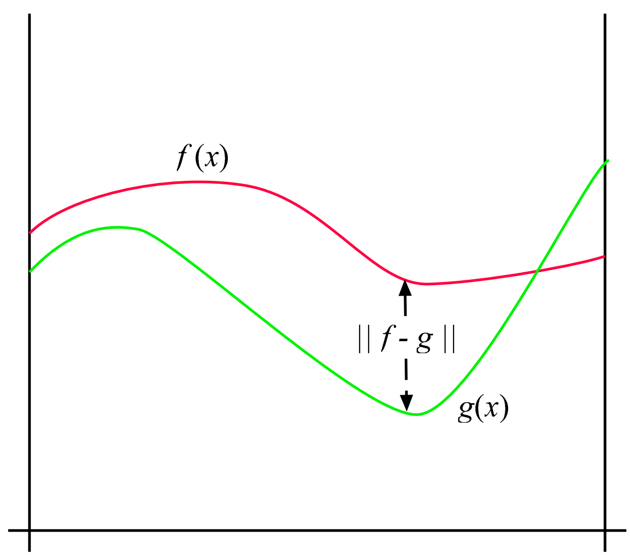
The *norm* measures how far apart two functions are. So,

$$||f - g|| = \text{ distance between } f \text{ and } g$$

Example

If f and g are in $C[a,b]$ then we define

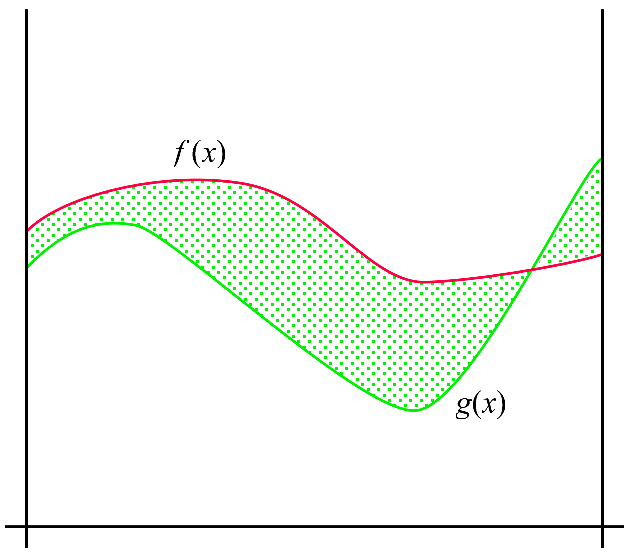
$$\|f - g\| := \max_{a \le x \le b} |f(x) - g(x)|$$



Another Example

If f and g are in $L^2[a,b]$ then we define

$$\|f-g\| := \left(\int_a^b |f(x)-g(x)|^2\,dx\right)^{\frac{1}{2}}$$



Banach Spaces

Both of these examples share an important technical property:

Let f_n be a list of vectors in a vector space (such as either of our last examples).

Suppose that the sequence of real numbers $a_n := ||f_n||$ satisfies:

$$\sum_{n=1}^{\infty} a_n \text{ converges}$$

Then the series of *vectors*, $\sum_{n=1}^{\infty} f_n$ also converges (to a vector).

Here's Our Main Application

Consider the first order differential equation,

$$\frac{dy}{dx} = F(x, y)$$

with initial condition

$$y(x_0) = y_0.$$

We'd like to know if there is a solution to this initial value problem on the interval $[a, b]$. And if there is a solution, is it the only one?

Assumptions

We'll assume $F(x, y)$ is continuous on the rectangle:

$$R := \{(x, y) \ : \ a \le x \le b, c \le y \le d\}$$

Lipschitz Condition

We'll also assume that there's a constant K so that:

$$|F(x, y_1) - F(x, y_2)| \le K|y_1 - y_2|$$

for all $a \le x \le b$ and $c \le y_1, y_2 \le d$.

Example

Consider

$$\frac{dy}{dx} = x^2 + y^2$$

with

$$y = 1 \text{ when } x = 0$$

Restricting attention to $-10 \le x \le 10$ and $-10 \le y \le 10$, we see that $F(x, y) := x^2 + y^2$ satisfies the Lipschitz Condition:

$$\begin{aligned} |F(x, y_1) - F(x, y_2)| &= |y_1^2 - y_2^2| \\ &= |y_1 + y_2| \, |y_1 - y_2| \\ &\le 20|y_1 - y_2| \end{aligned}$$

Goal

We want to find a function f so that;

$$f'(x) = F(x, f(x))$$

and

$$f(x_0) = y_0$$

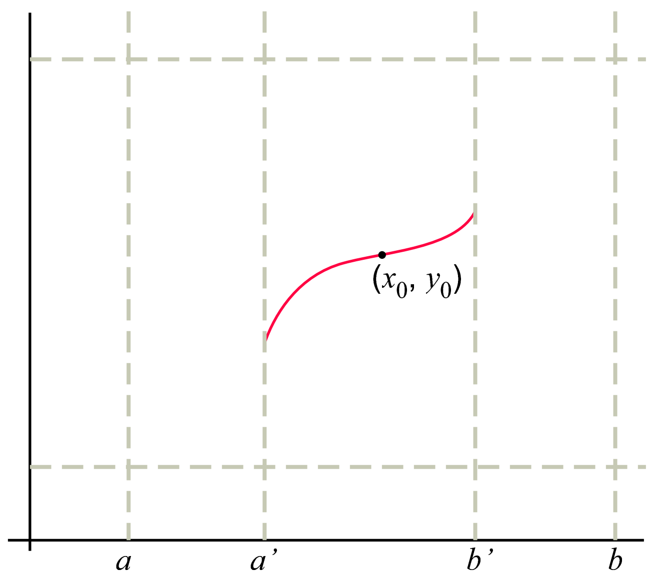
Picard's Theorem

We can find a subinterval $[a', b']$ of $[a, b]$ (also containing x_0) on which there is a function f such that:

$$f'(x) = F(x, f(x)) \text{ for all } a' \leq x \leq b'$$

and such that

$$f(x_0) = y_0$$



(x_0, y_0)
a
a'
b'
b

Example Revisited

This says there is a unique solution $y = f(x)$ such that

$$\frac{dy}{dx} = x^2 + y^2$$

and

$$f(0) = 1,$$

at least on an interval

$$a \leq x \leq b$$

where

$$-10 < a < 0 \text{ and } 0 < b < 10$$

Goal

We want to find a function f so that;

$$f'(x) = F(x, f(x))$$

and

$$f(x_0) = y_0$$

Idea

Write

$$f(x) - f(x_0) = \int_{x_0}^{x} f'(t)\,dt = \int_{x_0}^{x} F(t, f(t))\,dt$$

and so

$$f(x) = y_0 + \int_{x_0}^{x} F(t, f(t))\,dt$$

The Operator

Focus on the transformation:

$$g \longrightarrow y_0 + \int_{x_0}^{x} F(t, g(t))\, dt$$

In other words, for any function g, write $T(g)$ for the function h where

$$h(x) := y_0 + \int_{x_0}^{x} F(t, f(t))\, dt$$

So this means

We need to find a function f so that:

$$T(f) = f$$

The Contraction Mapping Theorem

Let S be a closed subset of a Banach space and let T be a function that maps S back into itself.

Suppose also that there's a number a, which is smaller than 1, and such that

$$||T(p) - T(q)|| \leq a||p - q||$$

(Anything with this property is called a contraction.)

Then there is a single point P in S at which

$$T(P) = P$$

Why?

Pick a point (any point) and call it p_0. Then look at

$$\begin{aligned} p_1 &:= T(p_0) \\ p_2 &:= T(p_1) = T(T(p_0)) \\ p_3 &:= T(p_2) = T(T(p_1)) = T(T(T(p_0))) \end{aligned}$$

and in general

$$p_n := T(p_{n-1}) = T(T(T(\cdots T(p_0)\cdots)))$$

Suppose the sequence converges

Let $P := \lim_{n\to\infty} p_n$. This means;

$$p_0 \to p_1 \to p_2 \to p_3 \to \cdots \quad P$$

and so

$$T(p_0) \to T(p_1) \to T(p_2) \to T(p_3) \to \cdots \quad T(P)$$

But how do we know they don't just bounce around?

$$\begin{aligned} \|p_n - p_{n-1}\| &= \|T(p_{n-1}) - T(p_{n-2})\| \\ &\leq a\|p_{n-1} - p_{n-2}\| \end{aligned}$$

$$\begin{aligned} \|p_n - p_{n-1}\| &\leq a\|p_{n-1} - p_{n-2}\| \\ &\leq a^2\|p_{n-2} - p_{n-3}\| \\ &\leq \cdots \\ &\leq a^n\|p_1 - p_0\| \end{aligned}$$

So what?

So

$$\sum_{n=1}^{\infty} \|p_n - p_{n-1}\| \text{ converges}$$

by term-by-term comparison with the series

$$\|p_1 - p_0\| \sum_{n=1}^{\infty} a^n$$

And so (by the properties of Banach spaces!), the following series converges;

$$\begin{aligned} \sum_{n=1}^{\infty} p_n - p_{n-1} &= \lim_{k\to\infty} \sum_{n=1}^{k} p_n - p_{n-1} \\ &= \lim_{k\to\infty} p_k - p_0 \end{aligned}$$

Telescoping Sums

$$
\begin{aligned}
& p_n - p_{n-1} \\
& \quad + p_{n-1} - p_{n-2} \\
& \qquad + p_{n-2} - p_{n-3} \\
& \qquad\quad + \cdots \\
& \qquad\qquad + p_1 - p_0
\end{aligned}
$$

Where Were We?

For any function g, we wrote $T(g)$ for the function h where

$$h(x) := y_0 + \int_{x_0}^{x} F(t, f(t))\, dt$$

So, is this a contraction?

Probably Not...

$$\|T(f) - T(g)\| = \max_{a \le x \le b} \left| \int_{x_0}^{x} F(t, f(t)) - F(t, g(t))\, dt \right|$$

Remember

$$|F(t, y_1) - F(t, y_2)| < K|y_1 - y_2|$$

and so

$$\|T(f) - T(g)\| \le \max_{a \le x \le b} K \int_{x_0}^{x} |f(t) - g(t)|\, dt$$

and

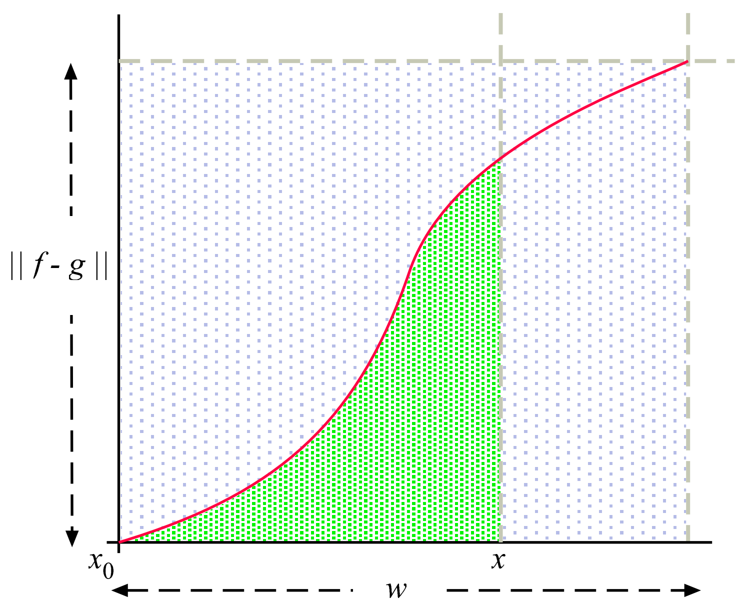
$$\begin{aligned} \max_{a \le x \le b} K \int_{x_0}^{x} |f(t) - g(t)|\, dt &\le Kw \max |f(t) - g(t)| \\ &= Kw\, \|f - g\| \end{aligned}$$

where

$$w := \max |x - x_0|$$

Why is

$$\int_{x_0}^{x} |f(t) - g(t)| \, dt \leq Kw \, \|f - g\|$$



But...

Take

$$\begin{aligned} a' &= x_0 - \frac{1}{2K} \\ b' &= x_0 + \frac{1}{2K} \end{aligned}$$

So that if we restrict to $[a', b']$ then everything is the same, except now

$$w = \frac{1}{2K}$$

so that

$$||T(f) - T(g)|| \leq \frac{1}{2}||f - g||$$